求极限值的若干方法

一、直接求极限.

☆1、等价代换.

$x\to 0$ $\quad x\sim \sin x\sim \tan x\sim \arctan x$

$\sim \arcsin x\sim \ln(1+x)\sim e^x-1$

① $\lim\limits_{n\to\infty}\dfrac{n^3\sqrt[n]{2}\,(1-\cos\frac{1}{n^2})}{\sqrt{n^2+1}-n}$

$=\lim\limits_{n\to\infty}\dfrac{n^2(1-\cos\frac{1}{n^2})}{\sqrt{1+\frac{1}{n^2}}-1}=\lim\limits_{n\to\infty}\dfrac{n^2\cdot\frac{1}{2}\cdot\frac{1}{n^4}}{\frac{1}{2}\cdot\frac{1}{n^2}}=1$

② $\lim\limits_{x\to 0}\dfrac{x(1-\cos x)}{(1-e^x)\sin x^2}=\lim\limits_{x\to 0}\dfrac{x\cdot\frac{1}{2}x^2}{-x\cdot x^2}=-\dfrac{1}{2}$

③ $\lim\limits_{x\to 0^+}\ln(1+\frac{1}{x})(e^x-1)=0$

2. 初等变形

① $\lim\limits_{n\to\infty}\left(\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{17}{16}\cdots\cdots\frac{2^{2^n}+1}{2^{2^n}}\right)$

$2(1-\frac{1}{2})(1+\frac{1}{2})(1+\frac{1}{2^2})(1+\frac{1}{2^4})\cdots(1+\frac{1}{2^{2^n}})$

$=2\cdot(1-\frac{1}{2^{2^{n+1}}})\quad\to 2$

② $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\dfrac{1}{\sqrt{1^3+2^3+\cdots+i^3}}=\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\dfrac{1}{\frac{1}{2}i(i+1)}$

$=\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}2\left(\frac{1}{i}-\frac{1}{i+1}\right)=2$

☆3. 指对转化

$a_1,\cdots a_n>0$

① $\lim\limits_{x\to 0}\left[\dfrac{a_1^x+\cdots+a_n^x}{n}\right]^{\frac{1}{x}}=\sqrt[n]{a_1\cdots a_n}$

② 若 x_n 收敛, $x_n>0$

$\lim\limits_{n\to\infty}\sqrt[n]{x_1\cdots x_n}=\lim\limits_{n\to\infty}x_n$

☆4. 利用Euler常数.

$\lim\limits_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n\right)=C$

求 $\lim\limits_{n\to\infty}\left(\frac{1}{n+1}+\cdots+\frac{1}{2n}\right)=\ln 2$

☆5. 变量替换

若 $\lim\limits_{n\to\infty}a_n=a$, $\lim\limits_{n\to\infty}b_n=b$

$\lim\limits_{n\to\infty}\dfrac{x_1y_n+x_2y_{n-1}+\cdots+x_ny_1}{n}=ab$

☆6. 两边夹法则

① $\lim\limits_{n\to\infty}\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdots\cdots\frac{2n-1}{2n}=0$

② $\lim\limits_{p\to\infty}\left[\left(\sum\limits_{i=1}^{n}a_i^p\right)^{\frac{1}{p}}+\left(\sum\limits_{i=1}^{n}a_i^{-p}\right)^{\frac{1}{p}}\right]$

$=\max\{a_1\cdots a_n\}+\max\{\frac{1}{a_1}\cdots\frac{1}{a_n}\}$

☆7. L Hospital 法则

① $\lim\limits_{x\to 0}\left(\dfrac{1}{\ln(1+x)}-\dfrac{1}{x}\right)=\lim\limits_{x\to 0}\dfrac{x-\ln(1+x)}{x\ln(1+x)}=\dfrac{1}{2}$

② $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{1-\cos x}}=\lim\limits_{x\to 0}e^{\frac{\ln(\frac{\sin x}{x})}{1-\cos x}}$

$\lim\limits_{x\to 0}\dfrac{(\ln\frac{\sin x}{x})'}{(\frac{x^2}{2})'}=\lim\limits_{x\to 0}\dfrac{\frac{\cos x}{\sin x}-\frac{1}{x}}{x}=-\dfrac{1}{3}$

★ Taylor 公式

$(1+x)^\alpha = 1+\alpha x + \frac{\alpha(\alpha-1)}{2!} + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + o(x^n)$

① $\lim\limits_{x\to 0} \frac{\frac{x^2}{2}+1-\sqrt{1+x^2}}{(\cos x - e^{x^2})\sin x^2}$

$= \lim\limits_{x\to 0} \frac{\frac{x^2}{2}+1-1-\frac{1}{2}x^2-\frac{\frac{1}{2}(-\frac{1}{2})}{2}\cdot x^4+o(x^4)}{(1-\frac{1}{2}x^2+\frac{1}{4!}x^4+o(x^4))-(1+x^2+o(x^2))\cdot x^2}$

$\cos x = 1-\frac{1}{2!}x^2+\frac{1}{4!}x^4 + \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + o(x^{2n})$

$= \lim\limits_{x\to 0} \frac{\frac{1}{8}x^4+o(x^4)}{(-\frac{3}{2}x^2+o(x^2))x^2}$

$= -\frac{1}{12}$

② $\lim\limits_{n\to\infty} n\left[e-(1+\frac{1}{n})^n\right]$

$= \lim\limits_{n\to\infty} n\left[e-e^{n\ln(1+\frac{1}{n})}\right]$

$= \lim\limits_{n\to\infty} \frac{e(1-e^{n\ln(1+\frac{1}{n})-1})}{\frac{1}{n}}$

$= \lim\limits_{n\to\infty} \frac{-e(n\ln(1+\frac{1}{n})-1)}{\frac{1}{n}}$

$= \lim\limits_{n\to\infty} \frac{-e(n(\frac{1}{n}-\frac{1}{2}\frac{1}{n^2}+o(\frac{1}{n^2}))-1)}{\frac{1}{n}}$

$\ln(1+x) = x-\frac{1}{2}x^2+\frac{1}{3}x^3+\cdots+\frac{(-1)^{n-1}}{n}x^n+o(x^n)$

$= \lim\limits_{n\to\infty} \frac{-e(-\frac{1}{2n}+o(\frac{1}{n}))}{\frac{1}{n}}$

$= \frac{e}{2}$

Else,

$\lim\limits_{n\to\infty} \sin^2(\pi\sqrt{n^2+n})$

$= \lim\limits_{n\to\infty} \sin^2(\pi\sqrt{n^2+n}-n\pi)$

$= \lim\limits_{n\to\infty} \sin^2\left(\pi\frac{1}{\sqrt{1+\frac{1}{n}}+1}\right) \to 1$

* O. Stolz 公式

设 $\{x_n\}$ 严格递增，$\lim\limits_{n\to\infty} x_n = +\infty$

若 $\lim\limits_{n\to\infty} \frac{y_n-y_{n-1}}{x_n-x_{n-1}} = a$

则 $\lim\limits_{n\to\infty} \frac{y_n}{x_n} = a$

例 $\lim\limits_{n\to\infty} \frac{1^p+2^p+\cdots+n^p}{n^{p+1}} = \frac{1}{p+1}$

二、递推公式求极限．

① 单调有界原理．

$x_0 = 1$, $x_{n+1} = \sqrt{2x_n}$

$1 \le x_0 < 2$. $1\le x_n < 2$

$x_{n+1} \geq x_n$

$x_n \uparrow \Rightarrow x_n \to A$ $A=\sqrt{2A} \Rightarrow A=2$

② 映像压缩

$\{x_n\}$, $\exists\, q<1$. s.t.

$|x_{n+1}-x_n| \le q|x_n - x_{n-1}|$

则 $\{x_n\}$ 收敛

Proof: $|x_m - x_n| \le \sum\limits_{k=n+1}^{m} |x_k - x_{k-1}|$

$\le \sum\limits_{k=n+1}^{m} r^{k-1}|x_1-x_0|$

$= \frac{r^n-r^m}{1-r}|x_1-x_0|$

$\le \frac{r^n}{1-r}|x_1-x_0|$

$\xRightarrow{\text{Cauchy收敛准则}}$ $\{x_n\}$ 收敛．

例．$x_1 = 1$. $x_{n+1} = \frac{1}{1+x_n}$

函数的连续性

间断点 (概念、分类)

① $f(x) = \dfrac{\sin(x\pi)}{|x|(x+1)(x-\frac{1}{2})}$

0 跳跃
-1 可去
$\frac{1}{2}$ 第二类

② $f(x) = \dfrac{x^2-x}{x^2-1}\sqrt{1+\dfrac{1}{x^2}}$

$\dfrac{x}{x+1}\sqrt{1+\dfrac{1}{x^2}}$

$x=1$ $f(x)\to\frac{\sqrt{2}}{2}$ 可去

$x=-1$ 第二类

$x=0$ $f(x) = \dfrac{x}{x+1}\sqrt{\dfrac{1+x^2}{x^2}}$

$= \dfrac{x}{1+x}\dfrac{\sqrt{1+x^2}}{|x|}$

跳跃间断点

*③ $f(x) = \lim\limits_{t\to x}\left(\dfrac{\sin t}{\sin x}\right)^{\frac{x}{\sin t-\sin x}}$

导数

1、可导性 f 在 $U(x_0)$ 有定义.

$\lim\limits_{x\to x_0}\dfrac{f(x)-f(x_0)}{x-x_0}$ 存在

ex: $f\in C^2$, $f(a)=0$, $g(x)=\begin{cases}\dfrac{f(x)}{x-a} & x\neq a\\ A & x=a\end{cases}$

(1) 求 A, 使 $g(x)$ 连续

(2) $g'(x)$ (3) $g'(x)$ 在 a 处连续

(1) $A=f'(a)$

(2) $g'(x)=\begin{cases}\dfrac{f'(x)(x-a)-f(x)}{(x-a)^2} & x\neq a\\ \frac{1}{2}f''(a) & x=a\end{cases}$

$$\lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}-f'(a)}{x-a} = \lim_{x\to a}\frac{f(x)-f(a)-f'(a)(x-a)}{(x-a)^2}$$

$$= \lim_{x\to a}\frac{f'(x)-f'(a)}{2(x-a)}$$

$$= \frac{1}{2}f''(a)$$

(3) $\lim\limits_{x\to a}g'(x) = \lim\limits_{x\to a}\dfrac{f'(x)(x-a)+f(x)-f'(x)}{2(x-a)}$

$= \lim\limits_{x\to a}\dfrac{f''(x)}{2} = \dfrac{f''(a)}{2}$

2. 隐函数求导

① $F(x,y)=0$

ex, $xy^2+1=ye^x$ $x=0\Rightarrow y(0)=1$

$2xyy'+y^2=(y+y')e^x$ $x=0\Rightarrow 1=y(0)+y'(0)$

$y'(0)=0$

$2yy'+2x(y'^2+y\cdot y'')+2y\cdot y'=(y+2y'+y'')e^x$

$x=0\Rightarrow y(0)+2y'(0)+y''(0)=0$

$y''(0)=-1$

② $\begin{cases}x=\varphi(t)\\ y=\psi(t)\end{cases}$

ex: $\begin{cases}x=\ln(1+t^2)\\ y=t-\arctan t\end{cases}$

$\dfrac{dy}{dx}=\dfrac{1-\frac{1}{1+t^2}}{\frac{2t}{1+t^2}}=\dfrac{t}{2}$

$\dfrac{d^2y}{dx^2}=\dfrac{d(\frac{t}{2})}{dt}\cdot\dfrac{1}{\frac{dx}{dt}}=\dfrac{1}{2}\cdot\dfrac{1}{\frac{2t}{1+t^2}}$

$=\dfrac{1+t^2}{4t}$

微分中值定理.

1. Fermat定理

f在x_0处取极值. 若$f'(x_0)$存在

则 $f'(x_0)=0$

2. 罗尔中值定理. $f(a)=f(b)$

3. Lagrange中值定理.

f在$[a,b]$连续. (a,b)上可导.

则 $\exists\ \xi\in(a,b)$ s.t.

$$f'(\xi)=\frac{f(b)-f(a)}{b-a}$$

4. 柯西中值定理.

f, F 在 $[a,b]$上连续, (a,b)可导

且 $F'(x)\neq 0$ 于(a,b), 则 $\exists\ \xi\in(a,b)$ s.t.

$$\frac{f(b)-f(a)}{F(b)-F(a)}=\frac{f'(\xi)}{F'(\xi)}$$

ex: f在$[0,1]$上连续, $(0,1)$上可导

且 $f(0)=0$, $\forall x\in(0,1)$ $f(x)\neq 0$, 证:

存在 $\xi\in(0,1)$ s.t.

$$\frac{nf'(\xi)}{f(\xi)}=\frac{f'(1-\xi)}{f(1-\xi)}$$

证: $F(x)=f^n(x)f(1-x)$ $\quad F(1)=F(0)=0$

$\exists\xi$

$0=F'(\xi)=nf^{n-1}(\xi)f'(\xi)f(1-\xi)-f^n(\xi)f'(1-\xi)$

$f^{n-1}(\xi)\neq 0$

$$\Rightarrow \frac{nf'(\xi)}{f(\xi)}=\frac{f'(1-\xi)}{f(1-\xi)}$$

ex: f在$[0,1]$上可导. $f(0)=f(1)$ $\quad f'(1)=1$

证: $\exists\ \xi\in(0,1)$, s.t. $f''(\xi)=2$.

证明: $f(0)=f(1)+f'(1)(0-1)+\frac{f''(\xi)}{2}(0-1)^2$

ex: f在$[a,b]$上连续, (a,b)上二阶可导.

$f(a)=f(b)\geq 0$, $\exists c$ $f(c)<0$.

证: $\exists\ \xi_1, \xi_2$ s.t.

$f''(\xi_1)>0$. $f''(\xi_2)>0$

证明: $f(x_0)$ 取最小值

$f'(x_0)=0$

$$f'(\eta_1)=\frac{f(b)-f(x_0)}{b-x_0}>0 \qquad \eta_1\in(x_0,b)$$

$$f'(\eta_2)=\frac{f(x_0)-f(a)}{x_0-a}<0 \qquad \eta_2\in(a,x_0)$$

$$f''(\xi_1)=\frac{f'(\eta_1)-f'(x_0)}{\eta_1-x_0}>0$$

$$f''(\xi_2)=\frac{f'(x_0)-f'(\xi_2)}{x_0-\xi_2}>0$$

ex: f在$[0,1]$上二阶可导. 且 $f(0)=f(1)=0$

(1) 若 $\max\limits_{0\leq x\leq 1}f(x)\cdot\min\limits_{0\leq x\leq 1}f(x)<0$,

证: $\exists\ \xi\in(0,1)$ s.t.

$f''(\xi)+f(\xi)=2f'(\xi)$

(2) 如果 $x\in(0,1)$. $f''(x)+f(x)\neq 2f'(x)$

证: f在$(0,1)$内无零点.

证: (1) $f''(\xi)-f'(\xi)-(f'(\xi)-f(\xi))=0$

$F(x)=e^{-x}f(x)$

$F(1)=F(0)=F(c)=0$

$\exists\ \xi_1\in(0,c)$, $\xi_2\in(c,1)$ s.t. $F'(\xi_1)=F'(\xi_2)=0$

$\Rightarrow \exists\ \xi\in(\xi_1,\xi_2)$ $F''(\xi)=0$

(2) 反证

Taylor公式

Lagrange型余项Taylor公式

f在x_0某邻域(a,b)有直到$(n+1)$阶导数.

则 $\forall x\in(a,b)$ 有

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}(\xi)}{\ }(x-x_0)^{n+1}$$

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Peano 佩亚诺型余项

若 f 在 x_0 处存在 n 阶导数,则

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + o((x-x_0)^n)$$

函数单调性与极值.

1、f 在 $[a,b]$ 连续, (a,b) 可导

(a,b) $f'(x) \geq 0$ $f(x)\uparrow$ 于 $[a,b]$

$f'(x) \leq 0$ $f(x)\downarrow$ 于 $[a,b]$

2、极值充分条件

f 在 x_0 处有二阶导,且 $f'(x_0)=0$, $f''(x_0)\neq 0$

(1) $f''(x_0) > 0$. f 在 x_0 处取极小值

(2) $f''(x_0) < 0$. f 在 x_0 处取极大值.

例: $f(x)$ 于 $x=0$ 点附近有定义,且

$$\lim_{x\to 0}\frac{\sin 2x + x f(x)}{x^3} = 0$$

(1) 求 $\lim_{x\to 0}\dfrac{2+f(x)}{x^2}$

(2) 证: 若 $f(x)$ 在 $x=0$ 处二阶可导,则 $x=0$ 为函数 $f(x)$ 极小值点.

证:(1) $\lim_{x\to 0}\dfrac{2+f(x)}{x^2} = \lim_{x\to 0}\dfrac{2x+xf(x)}{x^3}$

$= \lim_{x\to 0}\dfrac{\sin 2x + f(x) + (2x-\sin 2x)}{x^3}$

$= \lim_{x\to 0}\dfrac{2x-\sin 2x}{x^3}$

$= \lim_{x\to 0}\dfrac{2x-(2x-\frac{1}{6}(2x)^3+o(x^3))}{x^3}$

$= \dfrac{4}{3}$

(2) $\lim_{x\to 0}\dfrac{2+f(x)}{x^2} = \dfrac{4}{3}$

$f(0) = -2.$

$= \lim_{x\to 0}\dfrac{f'(x)}{2x} = \dfrac{4}{3}$

$f'(0) = 0$

$= \lim_{x\to 0}\dfrac{f''(0)}{2} = \dfrac{4}{3}$

$f''(0) = \dfrac{8}{3} > 0$

0 为最小值点

* 设 f 在 $(-\infty,+\infty)$ 三阶可导.

f, f''' 在 $(-\infty,+\infty)$ 上有界

证: f', f'' 在 $(-\infty,+\infty)$ 上有界

证明: $\forall x_0$

$$f(1+x_0) = f(x_0) + f'(x_0) + \frac{f''(x_0)}{2!} + \frac{f'''(\xi_1)}{3!}$$

$$f(-1+x_0) = f(x_0) - f'(x_0) + \frac{f''(x_0)}{2!} + \frac{f'''(\xi_2)}{3!}$$

$$\Rightarrow f''(x_0) = f(1+x_0) + f(-1+x_0) - 2f(x_0) + \frac{f'''(\xi_1)}{3!} - \frac{f'''(\xi_2)}{3!}$$

$$f'(x_0) = \frac{1}{2}\left(f(1+x_0) - f(1-x_0) - \frac{f'''(\xi_1)}{3!} - \frac{f'''(\xi_2)}{3!}\right)$$

$f'(x_0)$ $f''(x_0)$ 有界